Area of a General Polygon

By David Chandler

If you know the coordinates of the vertices of any polygon there is a very simple algorithm for calculating its area. Let's first look at the method, then why it works. Consider the 5-sided polygon shown here. Assume the measurements are in meters.

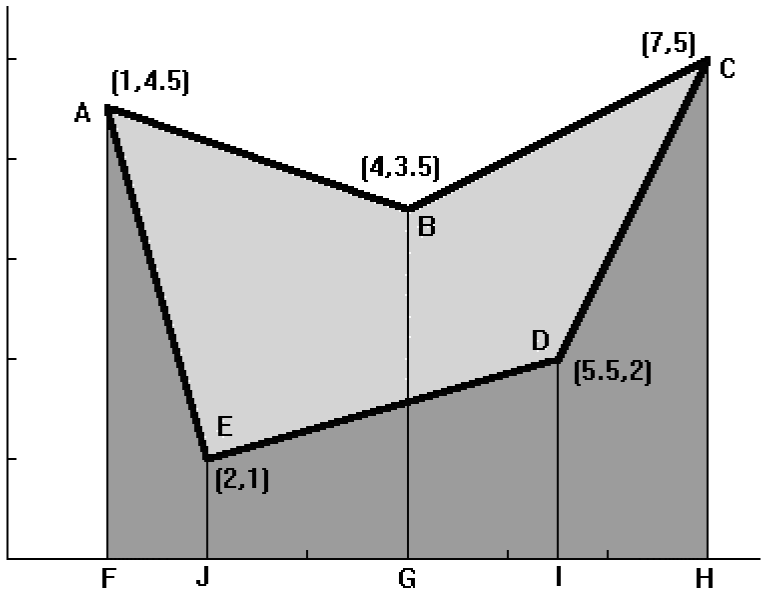

Fig. 1

List the coordinates, in order as you go around the polygon, *with the first point repeated as the last point to close the polygon*.

1	4.5
4	3.5
7	5
5.5	2
2	1
1	4.5

Now multiply all the diagonal numbers slanting down to the right putting the answers on the right ($1 \times 3.5 = 3.5$, $4 \times 5 = 20$, etc.), and all the diagonal numbers slanting to the left putting the answers on the left. (Use a calculator if the numbers get messy.)

	1	4.5	
18	4	3.5	3.5
24.5	7	5	20
27.5	5.5	2	14
4	2	1	5.5
1	1	4.5	9

Now add the two outer columns.

	1	4.5	
18	4	3.5	3.5
24.5	7	5	20
27.5	5.5	2	14
4	2	1	5.5
1	1	4.5	9
75			52

Finally, subtract the smaller from the larger sum and divide by 2.

$$\frac{75-52}{2}=11.5$$ **The area is 11.5 m²**

The method is exact; it works for any polygon; it is easy to remember; it is easy to carry out; and it is easy to implement on a spreadsheet for many-sided polygons and messy "real world" numbers.

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Now let's see why the method works. Let us re-label the polygon with general coordinates.

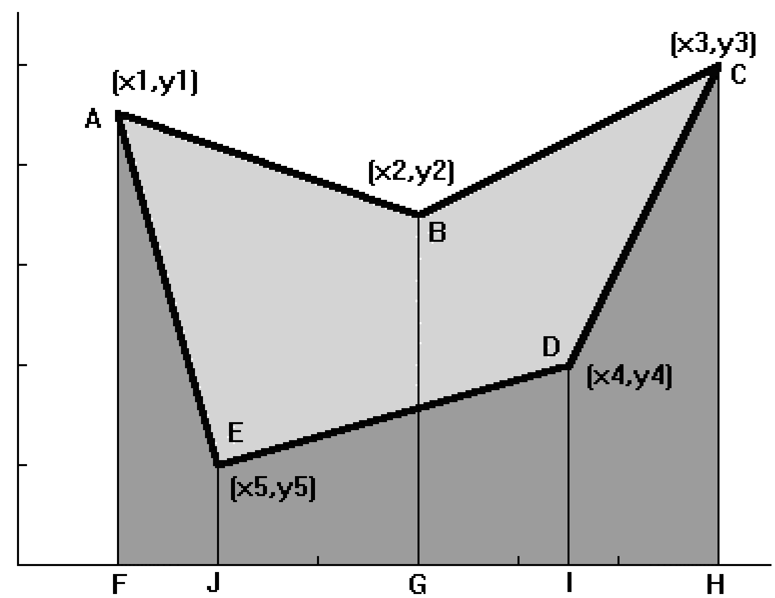

Fig. 2

The region under each edge of the polygon, projecting down to the x-axis, (for example, region ABGF) is a trapezoid. (A trapezoid is any four-sided figure with two opposite sides parallel.) The trapezoid under side AB is shown below:

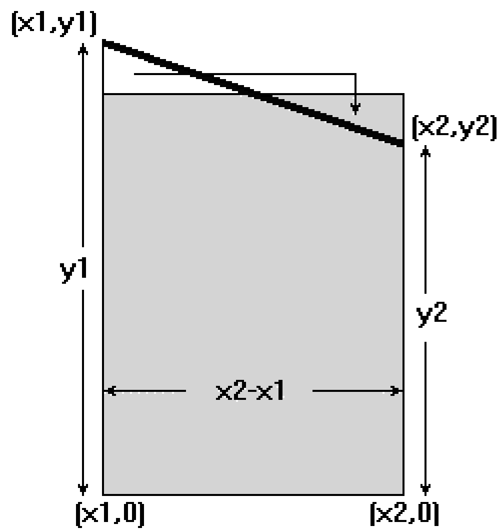

Fig. 3

If you clip off the top corner and paste it in as shown, you form a rectangle whose width is $(x_2 - x_1)$, and whose height is $\frac{(y_1 + y_2)}{2}$, the average of the two vertical sides of the trapezoid.

$$\text{Area of the shaded rectangle} = \text{Area of the trapezoid} = (x_2 - x_1)\frac{(y_1 + y_2)}{2}.$$

Notice that as long as we move from left to right, x_2 is larger than x_1, so $(x_2 - x_1)$ is positive. However, if we move from right to left, $(x_2 - x_1)$ will be negative. That's actually convenient. In our polygon (Fig. 2) we sweep from left to right, from A to B to C, adding up the areas of the first two trapezoids. Notice we have covered the whole polygon, but the calculation so far includes the area underneath as well. Now if we continue sweeping from right to left, from C to D to E and back to A, the calculation automatically gives us negative areas canceling the area *under* the polygon and leaving us with the area of the polygon itself.

We could plug numbers in and get our result right now, but we can simplify the formula by doing a little algebra first. Factor out the 2 and multiply the two binomials to get $\frac{1}{2}(x_2 y_1 + x_2 y_2 - x_1 y_1 - x_1 y_2)$ for the first term. Now repeat the pattern, shifting the subscripts, to get all five terms:

$$\frac{1}{2}\begin{bmatrix}(x_2y_1 + x_2y_2 - x_1y_1 - x_1y_2) + (x_3y_2 + x_3y_3 - x_2y_2 - x_2y_3) + \\ (x_4y_3 + x_4y_4 - x_3y_3 - x_3y_4) + (x_5y_4 + x_5y_5 - x_4y_4 - x_4y_5) + \\ (x_1y_5 + x_1y_1 - x_5y_5 - x_5y_1)\end{bmatrix}$$

Notice that $+x_2y_2$ is canceled by $-x_2y_2$. The same is true of all terms involving two *equal* subscripts. Removing the cancelled pairs leaves us with:

$$\tfrac{1}{2}\left[(x_2y_1 - x_1y_2) + (x_3y_2 - x_2y_3) + (x_4y_3 - x_3y_4) + (x_5y_4 - x_4y_5) + (x_1y_5 - x_5y_1)\right].$$

Grouping all the positive terms together and all the negative terms together we have:

$$\tfrac{1}{2}\left[(x_2y_1 + x_3y_2 + x_4y_3 + x_5y_4 + x_1y_5) - (x_1y_2 + x_2y_3 + x_3y_4 + x_4y_5 + x_5y_1)\right].$$

We have derived our algorithm! You multiply the diagonals one way and add them up, multiply the diagonals the other way and add them up, subtract the two results and divide by 2.

We have derived the area algorithm for a 5-sided polygon, but we could have repeated the pattern for any number of sides. The method also works even if the sides double back on themselves any number of times, as in Fig. 4. Draw a vertical line upward from any point in the region in or near the polygon. As we trace out the polygon we will cross *above* any point in the exterior of the polygon an even number of times (zero being an even number too), with an equal number of rightward and leftward sweeps. Regions containing such points will therefore cancel out of the computation. We will cross above any point in the interior of the polygon an odd number of times, with all but one pass canceling out. Depending on which direction we trace the polygon the answer may come out positive or negative, but dropping the sign we will always get the correct area.

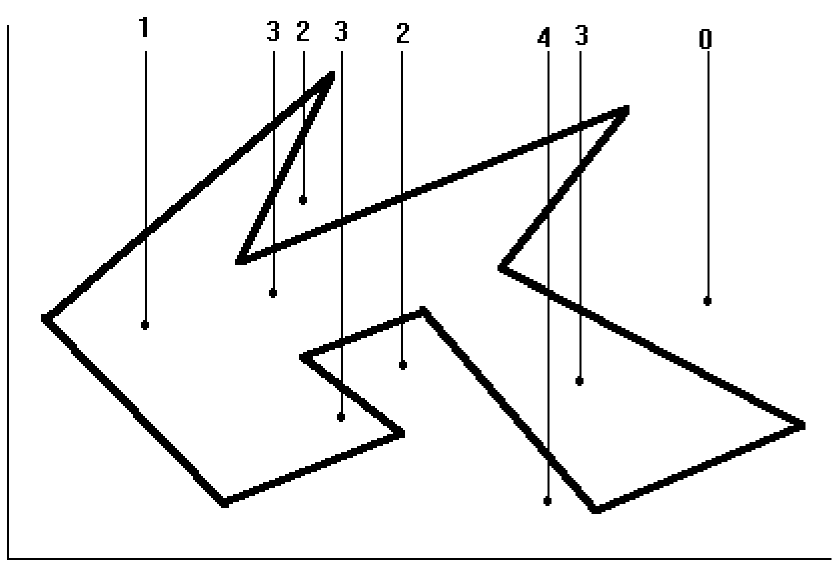

Fig. 4